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Notes on a semigroup related to the dicyclic group $\mathbf{Q_n}$

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Abstract

We consider certain properties of the semigroup S defined by the presentation

$$S = \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, ba = ab^{2^{n-1}-1}\rangle, \ (n \geq 3).$$

1 Introduction and Preliminary Facts

The purpose of this paper is to investigate computationally some remarkable properties of a certain finitely generated semigroup. For the terminology and notation see [4, 5]. We know that if A is an alphabet and A^+ denotes the free semigroup on A, then a *semigroup presentation* is a pair $\langle A : R\rangle$ where $R \subseteq A^+ \times A^+$. The elements of A are called *generators*, and the elements of R are *relations*. Some preliminaries and more information on semigroup presentations may be found in [3, 10]. However, there are many semigroup presentations that each of which has some specific properties [1, 10, 11].

The dicyclic group $\mathbf{Q_n}$ is given by the presentation

$$\langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, ba = ab^{2^{n-1}-1}\rangle,$$

where $n \geq 3$. We are interested here in the semigroup defined by the above presentation and so consider the following semigroup modification of it:

$$S = \langle a, b : a^{2^{n-1}+1} = a, b^2 = a^{2^{n-2}}, ba = ab^{2^{n-1}-1}\rangle, \quad (n \geq 3)$$

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For the semigroup S, some auxiliary algebraic properties can be verified inductively which we use throughout the paper. They show that the the semigroup S, as a non-group and non-commutative semigroup, is a concrete example of different kinds of semigroups.

Lemma 1.1. *For every $k \in \{0\} \cup \mathbb{N}$ with $n \geq 3$ we have:*

(a) if $i = 4k+1$ then $b^{(2^{n-1}-1)i} = a^{2^{n-2}}b$;

(b) if $i = 4k+2$ then $b^{(2^{n-1}-1)i} = a^{2^{n-2}}$;

(c) if $i = 4k+3$ then $b^{(2^{n-1}-1)i} = a^{2^{n-1}}b$;

(d) if $i = 4k+4$ then $b^{(2^{n-1}-1)i} = a^{2^{n-1}}$.

Proof. It is easy to see that modulo 2^{n-1} and for a positive integer t, the following trivial identities are satisfied:

$$t2^{n-1} - 2^{n-2} \equiv 2^{n-2} \qquad (1), \qquad t2^{n-1} - 2^{n-1} \equiv 2^{n-1}. \qquad (2)$$

We prove only the assertion concerning the part (a) and the remaining cases can be proved similarly. For (a) use an inductive method on k. Let $k = 0$ so we have:

$$\begin{aligned} b^{(2^{n-1}-1)} &= (b^2)^{(2^{n-2}-1)} \cdot b = \left(a^{2^{n-2}}\right)^{2^{n-2}-1} \cdot b \qquad \left(\text{for} \quad b^2 = a^{2^{n-2}}\right) \\ &= a^{(2^{2n-4}-2^{n-2})} \cdot b \overset{(1)}{=} a^{2^{n-2}} \cdot b. \end{aligned}$$

Assume (a) is true for k, i.e.; $b^{(2^{n-1}-1)(4k+1)} = a^{2^{n-2}}b$ then

$$\begin{aligned} a^{2^{n-2}} \cdot b &= a^{(2^{n-2}+2^{n-1})} \cdot b \overset{(2)}{=} a^{2^{n-2}}\left(a^{2^{n-2}}\right)^{(2^n-2)} \cdot b = a^{2^{n-2}} \cdot (b^2)^{(2^n-2)} \cdot b \\ &= (a^{2^{n-2}} \cdot b) \cdot b^{2^{n+1}-4} = b^{(2^{n-1}-1)(4k+1)} \cdot b^{2^{n+1}-4} = b^{(2^{n-1}-1)(4k+5)}. \end{aligned}$$

□

Lemma 1.2. *For $1 \leq i \leq 2^{n-1}$ we have $a^i = ba^{(i-1)2^{n-2}+i}b$.*

Proof. The result is true for $i = 1$. Indeed, $bab = (ab^{2^{n-1}-1}) \cdot b = ab^{2^{n-1}} = a(b^2)^{2^{n-2}} = a(a^{2^{n-2}})^{2^{n-2}} = a(a^{2^{2n-4}}) = a$. If the claim is true for i then the relations of S and the first part of Lemma 1.1 gives:

$$ba = ab^{2^{n-1}-1} = a(a^{2^{n-2}}b) = a^{2^{n-2}+1}b, \qquad (3)$$

and so $a^{i+1} = a^i \cdot a = (ba^{(i-1)2^{n-2}+i}b) \cdot a = (ba^{(i-1)2^{n-2}+i}) \cdot (ba)$ which by (3) is equal to

$$(ba^{(i-1)2^{n-2}+i}) \cdot (a^{2^{n-2}+1}b) = ba^{i2^{n-2}+(i+1)}b.$$

□

As a result of the above lemma we have:

Corollary 1.3. *In semigroup S we have:*

$$a^i b = ba^{i(2^{n-2}+1)},\ ba^i = a^{i(2^{n-2}+1)}b \quad (1 \leq i \leq 2^{n-1}).$$

Lemma 1.4. *The semigroup S may be partitioned as*

$$S = \{b\} \cup \{a^i, 1 \leq i \leq 2^{n-1}\} \cup \{ba^j, 1 \leq j \leq 2^{n-1}\}.$$

Proof. By the corollary above and the relations of S, we conclude that the only words in S starting with a are exact powers of a. □

Proposition 1.5. *For elements a and b while $1 \leq i, j \leq 2^{n-1}$ the following relations hold:*

(a) $(a^i)(b) = (b)(a^i) \cdot (ba^{2^{n-2}-i}) \cdot (b)(a^i)$;

(b) $(a^i)(a^j) = (a^j)(a^i) \cdot (a^{2^{n-2}-(i+j)}) \cdot (a^j)(a^i)$;

(c) $(a^i)(ba^j) = (ba^j)(a^i) \cdot (ba^{2^{n-2}(1+j)-(i+j)}) \cdot (ba^j)(a^i)$;

(d) $(b)(a^i) = (a^i)(b) \cdot (a^{2^{n-2}-i}b) \cdot (a^i)(b)$;

(e) $(b)(ba^i) = (ba^i)(b) \cdot (a^{2^{n-2}-i}) \cdot (ba^i)(b)$;

(f) $(b)(b) = (b)(b) \cdot (b^2) \cdot (b)(b)$;

(g) $(ba^j)(a^i) = (a^i)(ba^j) \cdot (ba^{(2^n-1)(i+j)+2^{n-2}}) \cdot (a^i)(ba^j)$;

(h) $(ba^i)(b) = (b)(ba^i) \cdot (ba^{2^{n-1}+i(2^{n-2}-2))}) \cdot (b)(ba^i)$;

(k) $(ba^i)(ba^j) = (ba^j)(ba^i) \cdot (a^{2^{n-2}(1+i)-(i+j)}) \cdot (ba^j)(ba^i)$;

(l) $(a^i)(a^{2^{n-1}-i}) = [(a^i)(a^{2^{n-1}-i})]^2$;

(m) $(b)(b^3) = [(b)(b^3)]^2$;

(n) $(ba^i)(ba^{i(2^{n-2}-1+2^{n-2}}) = [(ba^i)(ba^{i(2^{n-2}-1+2^{n-2}})]^2$.

Proof. We start from the right hand side of (a). Corollary 1.3 gives:

$$(b)(a^i)(ba^{2^{n-2}-i})(b)(a^i) = (a^{i(2^{n-2}+1)}b) \cdot (ba^{2^{n-2}-i}) \cdot (a^{i(2^{n-2}+1)}b)$$

which is equal to $a^{i\times 2^{n-2}+i+2^{n-2}+2^{n-2}-i+i\times 2^{n-2}+i}b$ and by (2) we get $a^i b$ as desired. For (c) Corollary 1.3 and the relations of S yield $(ba^j)(a^i) = a^{(1+2^{n-2})(i+j)}b$ and so

$$(ba^j)(a^i)(ba^{2^{n-2}(1+j)-(i+j)})(ba^j)(a^i) = a^i \cdot a^{j(2^{n-2}+1)}b = (a^i)(ba^j).$$

Rewriting the right hand side of (d) gives $(b)(a^{i(2^{n-2}+1)})(ba^i b)$ which is equal to $b(a^{i(2^{n-2}+1)})(a^{i(2^{n-2}+1)-i})$, which is the left part of (c). For (e) we have
$$\begin{aligned}(ba^i)(b)(a^{2^{n-2}-i})(ba^i)(b) &= (a^{i(2^{n-2}+1)+2^{n-2}})(a^{2^{n-2}-i})(a^{i(2^{n-2}+1)+2^{n-2}})\\ &= a^{i+2^{n-2}} = (b)(ba^i).\end{aligned}$$
Since by Corollary 1.3, $a^i(ba^j) = a^{i+j(2^{n-2}+1)}$, the right hand side of (g) can be simplified as

$$(ba^j)(a^{i(2^{n-2}+1)})\big(a^{2^{n-2}+1[(2^{n-2}-1)(i+j)+2^{n-2}]}\big)a^i(ba^j),$$

which is equal to

$$(ba^j)(a^{i(2^{n-2}+1)+2^{n-2}-(i+j)})(a^{2^{n-2}+1}b^2a^j) = (ba^j)(a^i).$$

Similarly $(ba^j)(ba^i) = a^{j\times(2^{n-2}+1)+2^{n-2}+i}$ by Corollary 1.3 and the right hand side of (k) is reduced as $a^{2^{n-2}+i\times 2^{n-2}+i+j} = (ba^i)(ba^j)$ which shows that (k) is valid. By using $(1), (2)$ and (3), the proofs for $(b), (l)$ and (n) are routine and considering the relations of S; $(f), (h)$ and (m) can be easily verified. $\square$

Proposition 1.6. *For every $x, y \in S$ we have*

$$xy = a^s, yx = a^r, \quad \textit{or} \quad xy = ba^s, yx = ba^r,$$

where $1 \le s, r \le 2^{n-1}$ and $r \equiv s$ modulo 2^{n-2}.

Proof. The proof is similar to the proof of Proposition 1.5 by taking possible forms of x and y of S. Firstly we note that, if $r \equiv s$ modulo 2^{n-2} then $s = r - k \times 2^{n-2}$ where $k \in \mathbb{Z}^+$ and so for an element $a \in S$ we have:

$$a^s = a^{r-k\times 2^{n-2}} = a^{r-k\times 2^{n-2}+k\times 2^{n-1}} = a^{r+k\times 2^{n-2}}.$$

And then, by the relations of S, all xy have forms a^s or ba^s where $1 \le s \le 2^{n-1}$ and none of them ends in b. When $xy = a^s$, we have the following possible cases:

(a) $x = a^i, y = a^j$;

(b) $x = ba^i, y = ba^j$;

(c) $x = b, y = b$;

(d) $x = b, y = ba^i$;

(e) $x = ba^i, y = b$.

In parts (a) and (c) we have $xy = a^{i+j} = yx$ and $xy = a^{2^{n-2}} = yx$ respectively so they are obviously satisfied. If $x = ba^i, y = ba^j$ where $1 \leq i, j \leq 2^{n-1}$ so by using Corollary 1.3 we have:

$$yx = a^s = a^{j(1+2^{n-2})+i+2^{n-2}+i\times 2^{n-1}} = a^r = xy \cdot a^{(i+j)2^{n-2}},$$

where $s = j(1+2^{n-2}) + i + 2^{n-2}$ and $r = i(1+2^{n-2}) + j + 2^{n-2} + (i+j)2^{n-2}$ respectively. Hence, $r \equiv s$ modulo 2^{n-2} and so (b) is true. The proof for parts (d) and (e) is similar and we check just part (d). Let $x = b$ and $y = ba^i$ for some $1 \leq i \leq 2^{n-1}$. Then we get $yx = a^{i(1+2^{n-2})+2^{n-2}} = a^{i+2^{n-2}} \times a^{i\times 2^{n-2}} = xy \times a^{i\times 2^{n-2}}$ which shows that the claim is valid for (d). Now, for elements $x, y \in S, xy = ba^s (1 \leq s \leq 2^{n-1})$. So we have the cases below:

(f) $x = a^i, y = ba^j$;

(g) $x = a^i, y = b$;

(h) $x = ba^j, y = a^i$;

(m) $x = b, y = a^i$.

The claims in parts (f) and (h) are proved similarly. The same is true when considering (g) and (m) so we need to check the validity of the proposition just in parts (f) and (g). For (f) we have $yx = ba^{i+j} = ba^r$ and $xy = ba^{r+i\times 2^{n-2}} = ba^s$ where $r \equiv s$ modulo 2^{n-2}. In (g) :

$$xy = a^i b = ba^{i(1+2^{n-2})} = ba^s, \quad yx = ba^i = ba^r,$$

which shows that $r \equiv s$ modulo 2^{n-2}. This completes the proof. □

Proposition 1.7. *In the semigroup S and for every elements x, y and z we have $xyzyx = yxzxy$.*

Proof. Let $x, y \in S$. According to the previous proposition, we can consider two cases for xy, i.e.; $xy = a^s$ ($yx = a^r$) or $xy = ba^s$ ($yx = ba^r$) where $1 \leq s \leq 2^{n-1}$ and $r \equiv s$ modulo 2^{n-2} . Suppose $z \in S$ and $xy = a^s$. If for some $1 \leq i \leq 2^{n-1}$, $z = a^i$ then $xyzyx = a^{s+i+r} = yxzxy$. If for some $1 \leq i \leq 2^{n-1}$, $z = ba^i$ then $xyzyx = a^s \cdot ba^i \cdot a^r = a^{r+k\times 2^{n-2}} \cdot ba^i \cdot a^r = a^r \cdot ba^{i+k\times 2^{n-2}(1+2^{n-2})} \cdot a^r = a^r \cdot ba^i \cdot a^s = yxzxy$ for some $k \in \mathbb{Z}^+$. If $z = b$ then for some $k \in \mathbb{Z}^+$ we get $xyzyx = a^s ba^r = a^{r+k\times 2^{n-2}} ba^r = a^r ba^{k\times 2^{n-2}(1+2^{n-2})} a^r = a^r ba^s$. For $z \in S$ and $xy = ba^s$, the proof is similar. $\square$

2 Main results

A semigroup S is called *commuting regular* if for any $x, y \in S$ there exists $z \in S$ such that $xy = yxzyx$. If for any $x \in S$ there exists $y \in S$ such that $xy = (xy)^2$, then S is called *E-inversive* [2]. Whenever for all x, y and z of S we have $xyzyx = yxzxy$ then S is known as a $C_2 - semigroup$ [9].

Theorem 2.1. *Let $n \geq 3$. The semigroup*

$$S = \langle a, b : a^{2^{n-1}+1} = a, b^2 = a^{2^{n-2}}, ba = ab^{2^{n-1}-1} \rangle,$$

is a finite non-abelian commuting regular and E-inversible semigroup of order $2^n + 1$. Moreover S is a $C_2 - semigroup$.

Proof. It is enough to consider different cases for $x, y \in S$ as in Lemma 1.4. Then, considering the results of Proposition 1.5 yields the proofs of commuting regularity and E-inversibility respectively. Obviously, S is a non-abelian semigroup of order $2^n + 1$. For the rest we consider Proposition 1.7. $\square$

Remark 1.
When $n = 3$, by Lemma 2.3. of [11], we showed that the semigroup:

$$S = \langle a, b : a^5 = a, b^2 = a^2, ba = ab^3 \rangle = \{a, b, a^2, a^3, a^4, ab, a^2b, a^3b, a^4b\}$$

is also a quasi-commutative semigroup of order 9.

Lemma 2.2. *For $n \geq 3$ all elements of S except for b are regular. Moreover S is a π-regular semigroup.*

Proof. The relations of S show b is an indecomposable element so it cannot be regular. For the other cases, we may consider the following points which can be verified easily:

$$a^i = a^i \cdot (a^{2^{n-1}-i}) \cdot a^i, \quad ba^i = ba^i \cdot (ba^{2^{n-2}(i+1)-i}) \cdot ba^i \qquad (1 \leq i \leq 2^{n-1})$$

Also, part (f) of Proposition 1.5 and the later equalities yield

$$b^2 \in b^2 S b^2, \quad x \in xSx,$$

for all $x(\neq b) \in S$. Therefore the semigroup S is π-regular. □

An idempotent $e \in S$ is called *primitive* whenever $f \in E(S)$ and $f = ef = fe$ then we have $f = e$. If in a semigroup S all idempotents are primitive then the semigroup is named *primitive*.

Theorem 2.3. *For $n \geq 3$, the only idempotent of S is $e = b^4$ and so S is primitive.*

Proof. As $(a^{2^{n-1}})^2 = a^{2^n} = a^{2^{n-1}(2^{n-1}+1)} = a^{2^{n-1}}$ where $n \geq 3$ so $a^{2^{n-1}} = b^4$ is an idempotent. Since b is not regular it cannot be an idempotent. This shows that:

$$(ba^i)^2 = a^{i(2^{n-1}+1)+2^{n-2}+i} \neq ba^i.$$

Indeed, $b \notin \langle a \rangle$. Therefore $E(S)$ is a singleton and so S is primitive. □

Corollary 2.4. *For $n \geq 3$, eSe is a unipotent monoid. In fact S is a unipotent semigroup and so it is a power joined semigroup.*

Proof. As a consequence of the previous theorem and Corollary 1 [2] eSe is a unipotent monoid. For the rest we may consider [7]. □

Lemma 2.5. *For $n \geq 3$, $S^2 = S - \{b\}$ is a unique proper maximal ideal of S. Moreover $S^2 = [a]$ in which $[a]$ is the principle ideal of S generated by a.*

Proof. Regarding Lemma 2.2 and that $S^2 \subsetneq S$, we have $S^2 = S - \{b\}$ so it is a maximal ideal of S. Obviously for any other proper maximal ideal N of S , $b \notin N$ and so $N \subseteq S^2$ and so $N = S^2$. By the identities 1.3 and Lemma 2.2, $S^2 \subseteq [a]$ and so the proof is complete. □

Remark 2. Since $b^2 \in S^2$ so, element b would be a nilpotent with respect to $[a]$ [6].

A regular semigroup S is called a *Clifford* semigroup if all idempotent elements of S are central [8].

Corollary 2.6. *For $n \geq 3$, S^2 is a Clifford semigroup.*

Proof. Obviously, $e = b^4 \in S^2$ is central. □

A semigroup S is *abundant* if every minimal ideal of S contains an idempotent element.

Corollary 2.7. *For $n \geq 3$, S is an abundant semigroup.*

Proof. Since every minimal ideal of S necessarily contains the only idempotent $b^4 \in S$ so the semigroup is abundant. $\square$

Lemma 2.8. *For $n \geq 3$, $S = [b]$ where $[b]$ is the principle ideal of S generated by b.*

Proof. Using Lemma 1.2 and identities 1.3 the proof is clear. $\square$

Corollary 2.9. *For $n \geq 3$, S has exactly two $\mathfrak{J}$ classes.*

Proof. Since $S^2 = S - \{b\}$ is a regular proper subsemigroup of S so

$$[b]_{\mathfrak{J}} \cap S = \{b\}, \qquad [a]_{\mathfrak{J}} \cap S = S^2 = [a].$$

$\square$

References

[1] A. Arjomandfar, C. M. Campbell and H. Doostie, *Semigroups related to certain group presentations*, Semigroup Forum **85** (2012), no. 3, 533–539.

[2] S. Bogdanovic, M. Ciric, A. Stamenkovic, *Primitive idempotents in semigroups*, Mathematica Moravica **5** (2001), 7–18.

[3] C. M. Campbell, E. F. Robertson, N. Ruskuc, R .M. Thomas, *Semigroup and group presentations*, Bull. London Math. Soc. **27** (1995), 46–50.

[4] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups I*, Amer. Math. Soc. (1961).

[5] J. M. Howie, *An introduction to semigroup theory*, Academic Press Inc, (1967).

[6] F. Kmet *On radicals in semigroups*, Math. Slovaca, **32**, no. 2 (1982), 183–188.

[7] R. G. Levin, T. Tamura, *Notes on commutative power joined semigroups*, Pacific J. Math., **35**, no. 3 (1970), 673–679.

[8] A. Nagy, *Special Classes of Semigroup*, Kluwer Academic Publishers, (2001).

[9] B. H. Neumann, T. Taylor, *Subsemigroups of Nilpotent Groups*, Proc. Roy. Soc. Ser. A, **274** (1963), 1–4.

[10] E. F. Robertson and Y. Unlu, *On semigroup presentations*, Proc. Edinburgh Math. Soc. **2** 36 (1993), no. 1, 55–68.

[11] M. R. Sorouhesh, H. Dosstie, *Quasi-commutative semigroups of finite order related to Hamiltonian groups*, Bull. Korean Math. Soc. **52** (2015), No. 1, 239–246.

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